SOME LESSONS WITH CUISENAIRE® RODS

The following pages provide an opportunity not only to show the material itself but to indicate methods of working with it that teachers and pupils could find useful.

Each situation, as it is displayed, can be explained and discussed. A considerable number of lessons can thus be derived from what is shown here. In particular this will be of interest to teachers who are newcomers to this material.

All the arrangements and examples were created from just one International Set of Cuisenaire® Rods, a set which contains 304 pieces.
All details available at www.cuisenaire.co.uk .

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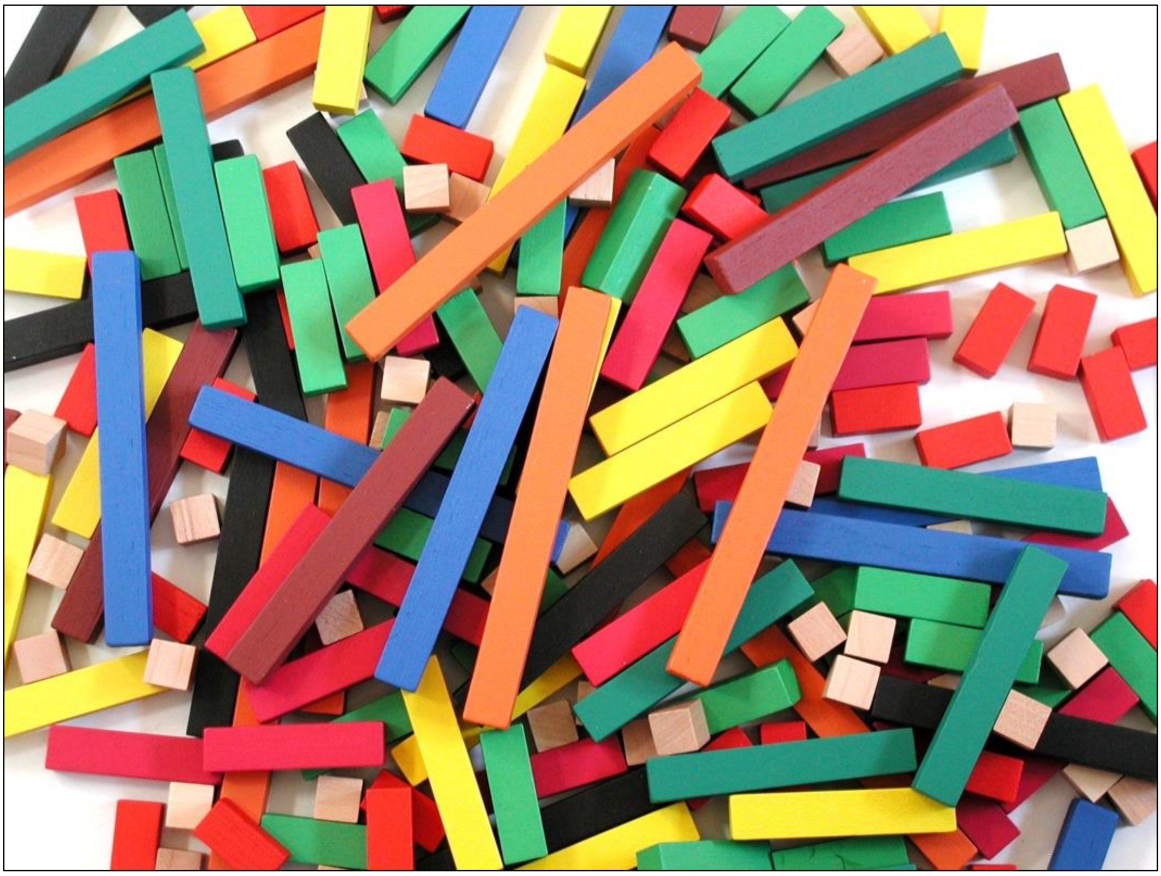
Contents

Description of the material

The Set of Cuisenaire® Rods



Situation 1 shows the contents of a set of Cuisenaire® rods.

The set is seen to contain cuboids in white, black, red, pink, tan, light green, dark green, blue, yellow and orange. Pupils examining the material discover that only these ten colours appear.

Find the elements of this set.

Viewers will notice the shape of each rod, which makes it possible to ask questions about the relative sizes of rods in the pile: which is the largest? which the smallest?

A more careful examination will reveal other characteristics.

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Equivalence by colours and length

Situation 2 shows groups of rods of the same colours.

Referring only to the colours of the rods, guess their order in increasing or decreasing length.

Passing from the one to the other of these first two situations makes clear the principle of **equivalence by colour**.

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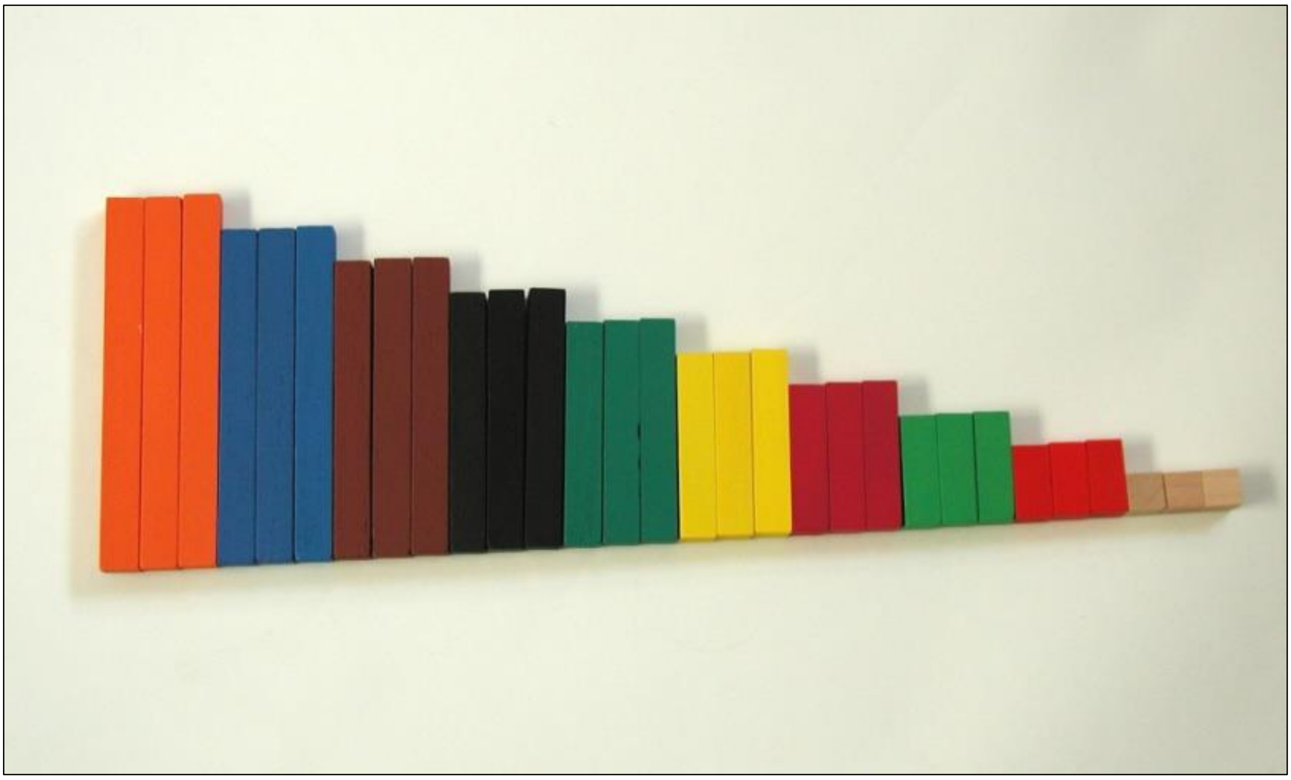
Order



Situation 3 gives the order, increasing in one direction and decreasing in the other, of the selection of rods above.

Thus if we take at random two rods a and b of the first set, it is clear that either a = b or a < b (a is smaller than or a > b (a is bigger than b).

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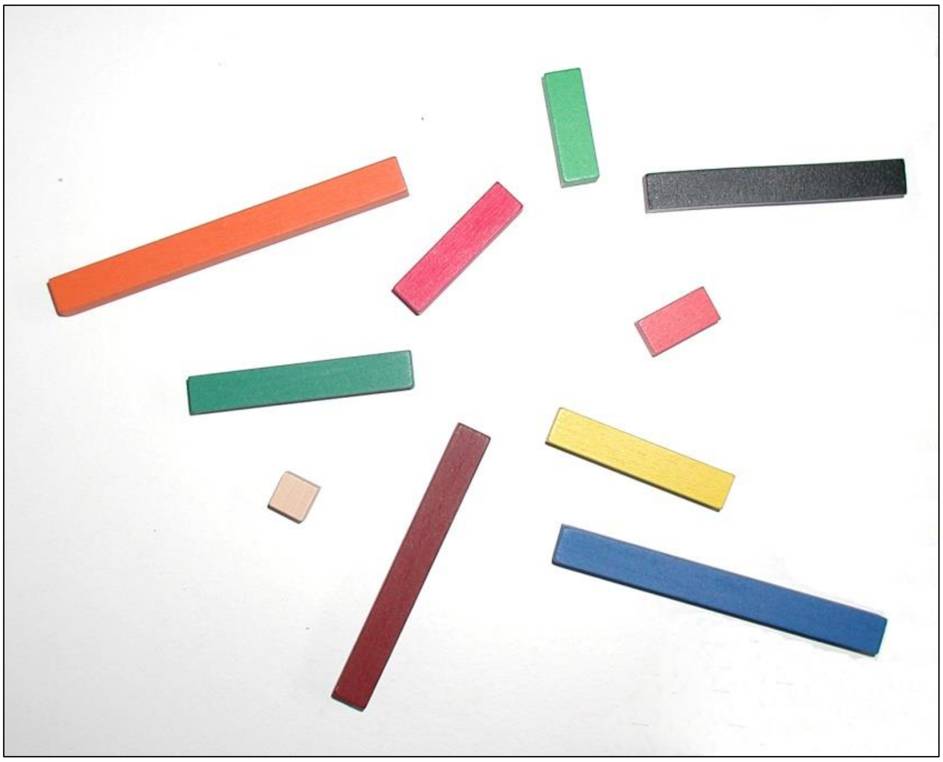
The quotient set



Situation 4. A single rod of each colour is enough to represent the set in a particular way.

To consider the set of equivalence classes (either by colour or by length) we can take one rod of each kind to represent each class.

This way of operating on the set produces an interesting sub-set which will be used several times as we proceed.

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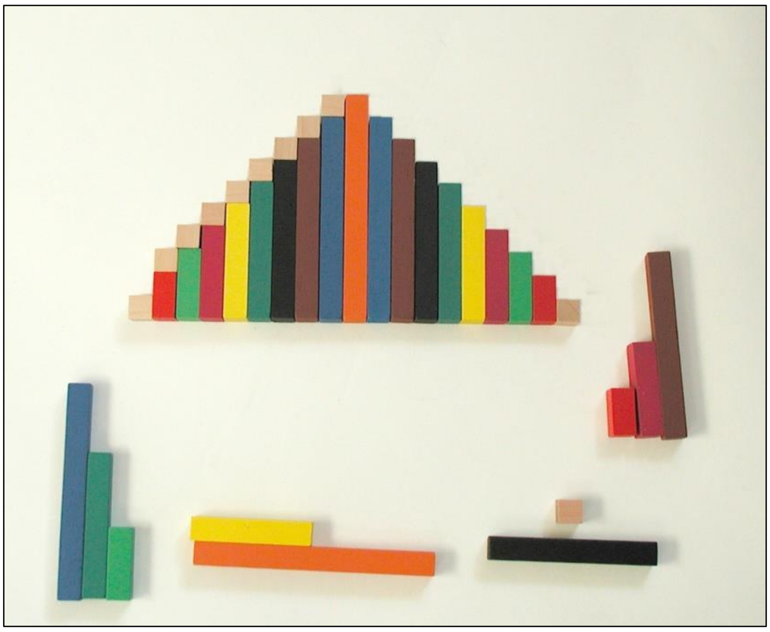
Characteristics of the quotient set



This situation deals with three samples of the sub-set above.

a) In the first example the staircase on the right shows simply that the sub-set can be ordered. But since no two rods in the set taken at random can be equal to each other, one inevitably being either smaller or larger than the other, it is said that the order is strict, or that the sub-set is strictly ordered; i. e. strictly increasing as we move from the white to the orange and strictly decreasing as we move from the orange to the white.

In the left-hand staircase formed by the sub-set a new property is revealed: that the difference between consecutive rods is always the same and is equal to the length of a white rod.

b) A third set has been arranged in sub-sets according to the principle of colour affinity. This way of putting the rods together gives what Georges Cuisenaire calls the 'families': the red, the yellow, the blue, the black and the white. The choice of colours and families is an essential part of Cuisenaire's invention.

Whilst making staircases your students may be asked to memorise the colours of the families in ascending and descending order. At least ten minutes may be needed for this. (For colour blind people this should be done by handling the rods, with recognition of the lengths to be associated with each colour.

The stairs with the white rods provide the opportunity for asking the following questions: How many white rods on top of each other do you think would be needed to produce the height of the red rod? of the light green rod? of the pink rod? of the yellow rod? etc. We can then return to the first staircase and give the names 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 to the rods, as measured by the white. The families are known by their colour names and by their numerals: (1); (7); (2, 4, 8); (3, 6, 9); (5, 10).

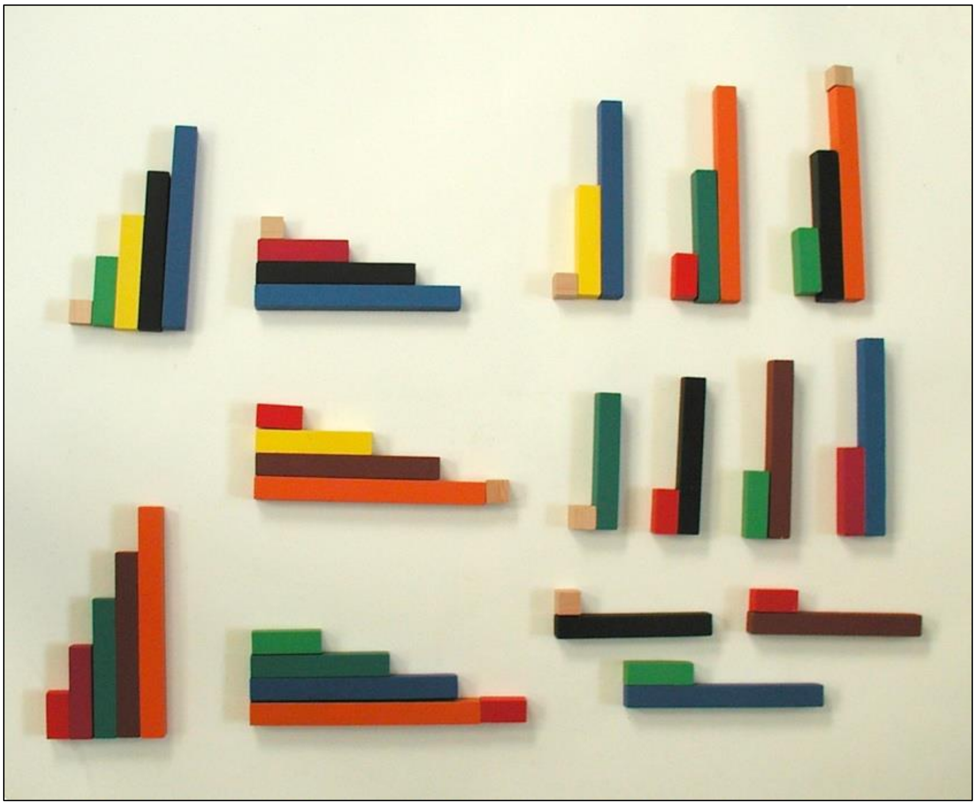
With this introduction to the material we have drawn attention to the fundamental structures:

- of sets
- of equivalence
- of classes of equivalence
- of order
- of ordered sets in two directions, strictly increasing and strictly decreasing
- and we have taught counting in both directions, from 1 to 10 and from 10 to 1.

STUDY OF WHOLE NUMBERS

Staircases

Counting is a rhythmic activity. There is thus a danger that the mind will wander when numbers are being recited. Saying them backwards and skipping some regular ones prevents the process from being automatic. (Situation 5 may be reconsidered here.)



This situation shows the first ten rods arranged in staircases so that the first step can be varied in height without altering the height of succeeding steps.

In particular these staircases show **odd** as well as **even** numbers.

All are **arithmetical progressions** and should be read in two directions. Some possible arithmetical progressions are contained in examples met already (for example, the series 2,3,4,5... is found within 1, 2, 3, 4 ...). Others which have been omitted for lack of space can be made with the rods.
See also in this connection Situation 28.

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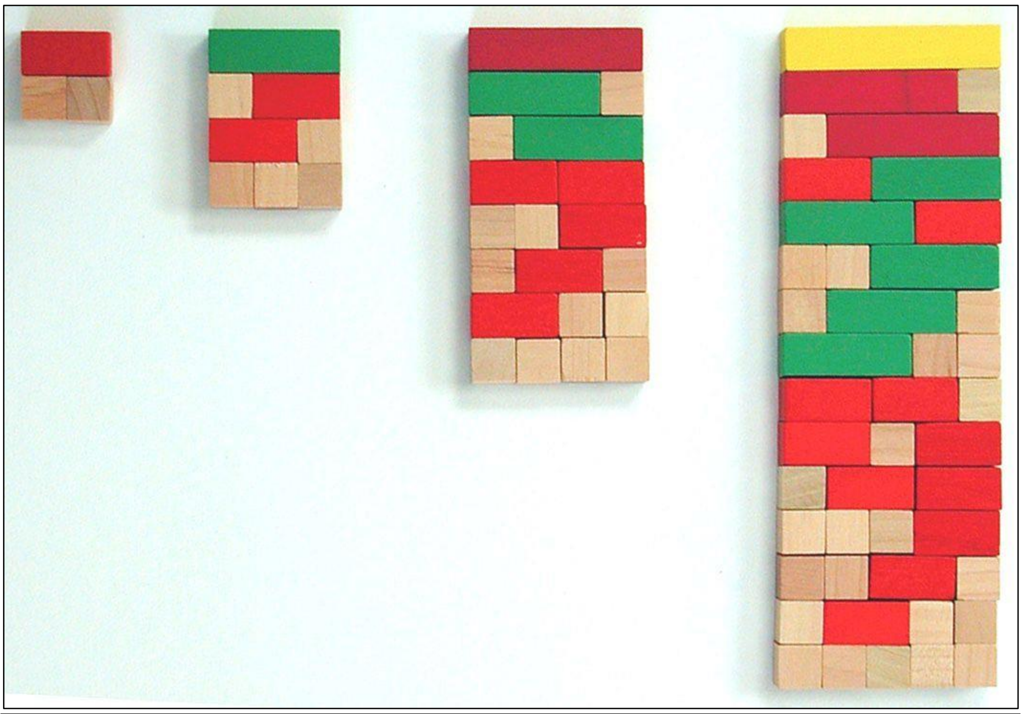
Decompositions



This situation contains tables of decompositions of the first rods. It shows:

a) that the number of decompositions increases very rapidly with the length of the rod:

2 for the red
4 for the light green
8 for the pink
16 for the yellow, and so on.

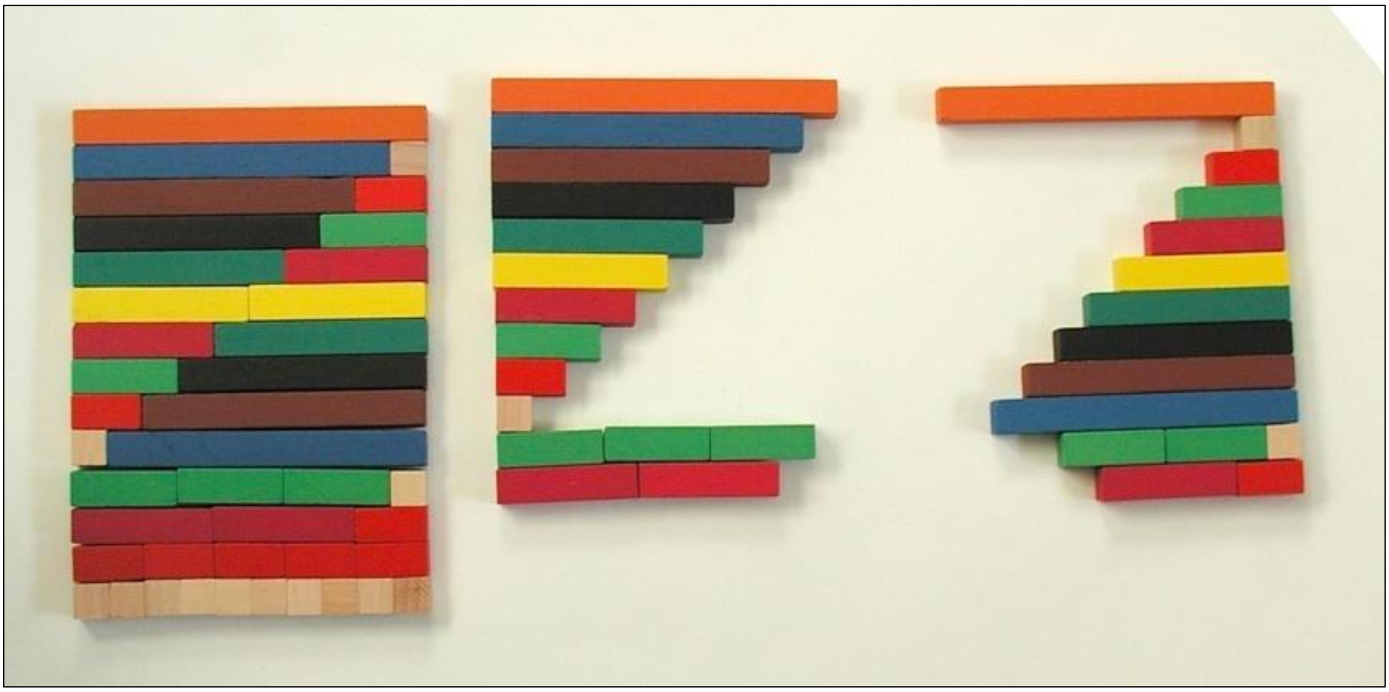
b) that each set of decompositions is what we can now name 2,3,4 or 5 respectively (which eliminates the danger that one colour only will always be associated with one number since 5, for example, can be represented as much by light green and red, or simply white, as it is by the yellow).

c) that the reading of each line is an addition, and that it is possible, using the notation of multiplication when one rod is repeated, to read into these tables of decomposition that

5=2 x 2 + 1
or
5 = 3 x 1 + 2
or
5 = 2 + 3 x 1 , etc.

* * *

Complementary sub-sets



This situation studies the decompositions of 10, first by means of two rods (giving the **complementary** sub-sets in a set having 10 elements), then using a larger number of rods in each line.

All lines of this table can also be read as divisions: e.g. 10 ÷ 3 and 10 ÷ 4.

But the essential point in this illustration is the associating of subtraction with the initial additions. When we find the rod that is needed to complete each line in order to reach a length equal to that of the rod in the first line, it is at the same time seen that the gesture can be made to the right or to the left (stressing that addition is commutative). On the other hand it is seen that if a number (or a length, or a rod) is given, every subtraction can be read as the inverse of an addition.

In fact more can be found: selecting one rod as the first line, we can mentally see each of the other rods beneath it, to the right or to the left. We can then ask:

what remains of the original rod when the part covered is equal to each of the other rods, taken in turn?

Here what is cut off is actually visible. But because of recognition of complements, it is evident that the mind sees the difference on the first rod; in fact, the difference is as easily seen as the amount cut off.

Using this link we can easily learn all subtractions up to 20 with the rods.

Those who use the rods will see how to develop the content of this situation step by step, thus achieving mastery of subtraction taught **simultaneously** with addition.

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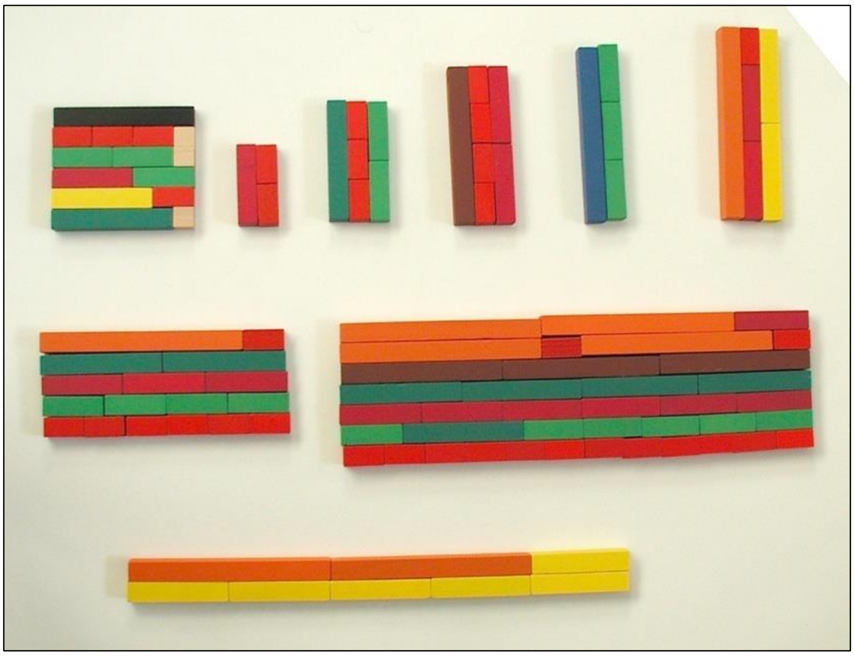
Situations 9-15 are concerned with multiplication, factors and rapid mental calculations.

Factors

The examples chosen are simple but the work they suggest can easily be generalised, especially if used in conjunction with the Product Chart invented by Georges Cuisenaire (shown in situation 11).

Lengths formed from one or more rods cannot all be made up by using rods of one colour only (excluding repetitions of the white rod) or by a chosen group of rods repeated a certain number of times. Those that cannot be so formed give the prime numbers. The others correspond to the numbers that have factors.



This situation shows with the first table of decompositions that 7 is prime. It also studies the factors of 4, 6, 8, 9, 10, 12, 24 and 25. It shows the lengths made up of rods of one colour immediately below the length corresponding to each number:

4	=	2	x	2						
6	=	2	x	3	=	3	X	2		
8	=	2	x	4	=	4	X	2		
9	=	3	x	3						
10	=	2	X	5	=	5	X	2		
12	=	2	X	6	=	6	X	2		
	=	3	X	4	=	4	X	3		
24	=	2	X	12	=	12	X	2		
	=	3	X	8	=	8	X	3		
	=	4	X	6	=	6	X	4		

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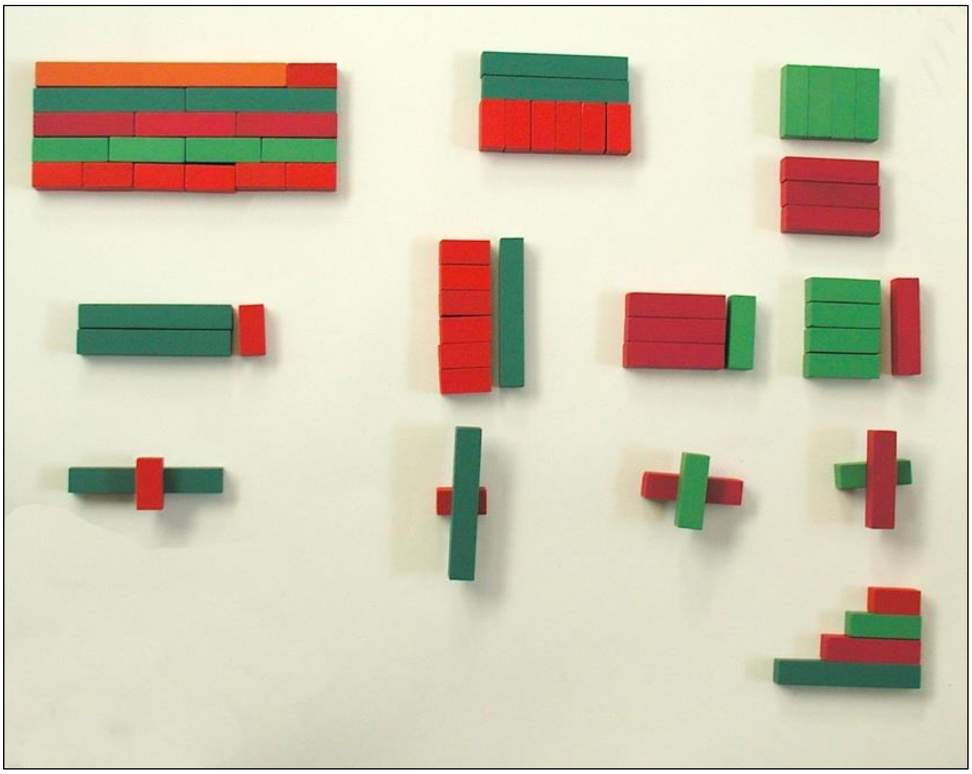
Composite Numbers

This situation shows how to pass from the table of decompositions of any **composite** number (the example here being 12) to the factors of that number, and conversely. The length 12, made up of 10+2, is also made up of 2 x 6, 3 x 4, 4 x 3 and 6 x 2.



Putting six red rods side by side and two dark green rods side by side, we form two rectangles that are found to be congruent on superimposing one on the other. Similarly, two rectangles made from four light green rods and from three pink rods are also congruent. The shortest dimension of the dark green rectangle is equal to the length of the red rod; the longest dimension of the red rectangle is equal to the length of the dark green rod. It can be seen from the frame what the other two arrangements of rods suggest.

So, if we represent with one cross of rods (in which the two dimensions are apparent) both the dark green and the red rectangles, or with another cross the light green and the pink rectangles, we can see how in each case we are also representing the original length orange and red end to end.

For us to return from a cross to the original length, it is enough to build the rectangles whose dimensions are shown, and to put the rods thus obtained end to end.

The cross is therefore a symbol for the **composite** number, since by following these instructions we can find that number.

The factors of twelve, i. e. the rods used in the crosses, are put on one side as something further that can be deduced from the study of this situation.

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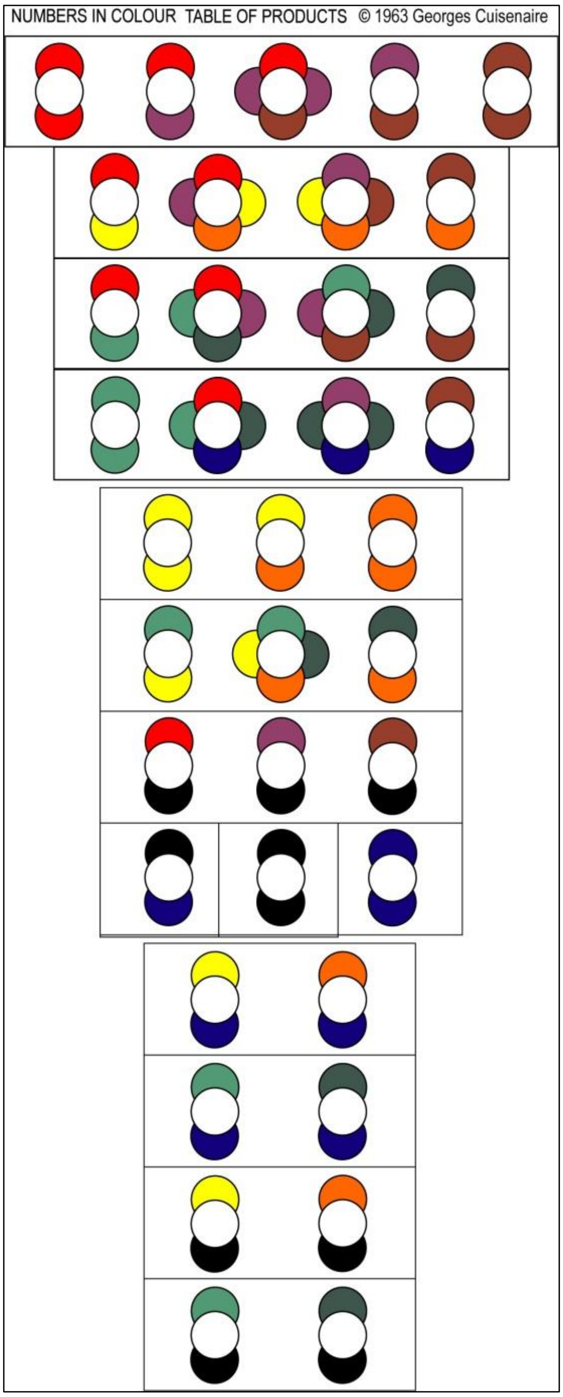
The Cuisenaire® Product Chart.



This shows the thirty-seven different products that can each be made by one or two crosses of rods.

But here we have the symbols of the products only.

A discussion on how to use the chart is found in the text Now Johnny Can Do Arithmetic.

The Product Chart and sets of Product Cards are important items in this method, and although mastery takes a little time, their use repays with considerable return for the time spent.

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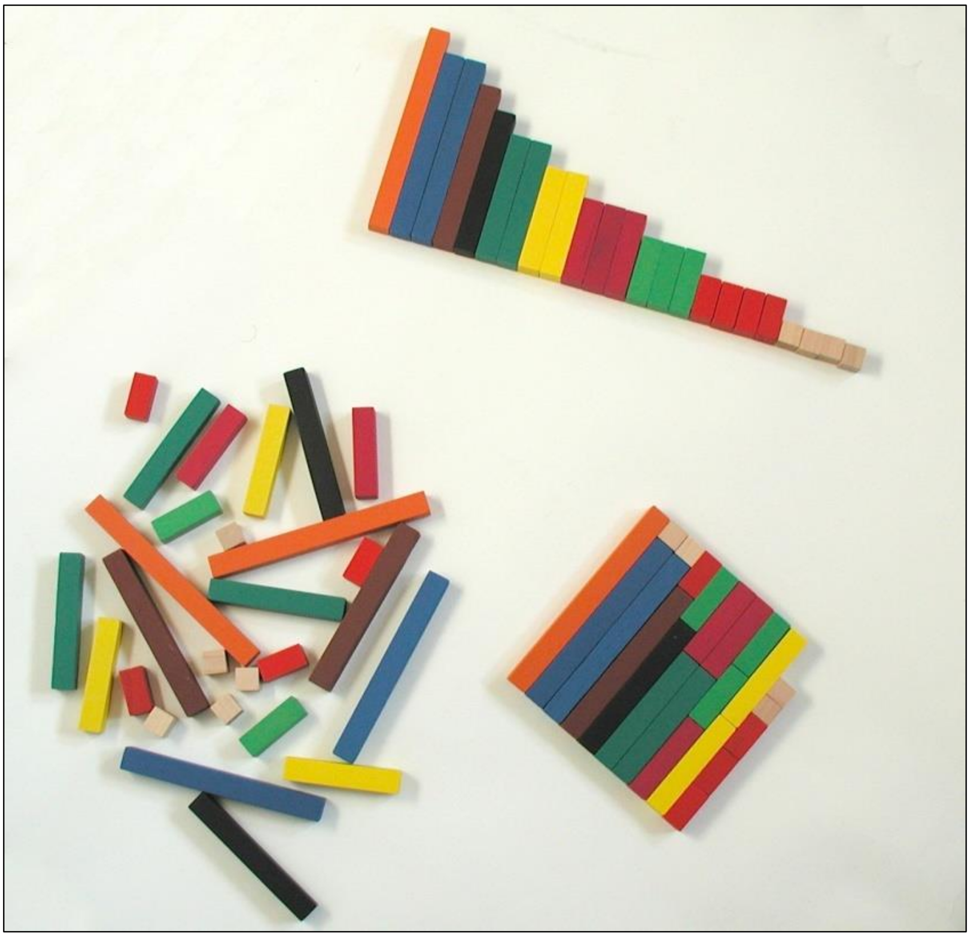
Mental Arithmetic on a handful of rods.

This situation assumes that products are being learnt and gives an interesting exercise. Though this may be repeated many times it retains its freshness since it presents each time a new challenge to the pupil and frees thought in a way not commonly known.



A handful of rods is placed before the pupils and they are told to find what length would be made if all the rods were put end to end. No one may touch them; they must operate mentally.
Two ways of operating are suggested:

a) In the first case the rods of one colour are noticed, counted and their product formed, this being remembered or added to what went before. In operating systematically from white to orange on all the colours that are present, the learners exhaust the pile. A count could be made of the number of operations involved and the time taken following this method, and of the number of errors that occur in using it.

b) The procedure that seems the most satisfactory to adopt is to look not for products to employ, but for complementary numbers forming tens. This is illustrated. The eye sweeps over the field and picks out the number of orange rods (if there are any) in the set, counting them and memorising the result. The choice of pairs is then made at random, black with light green, tan with red, dark green with pink, white with blue. Sometimes the positions of the rods suggest new combinations: for example, 3 tan rods and 1 dark green (equal to 3 orange rods) might be taken together to give 30. The variety is considerable.

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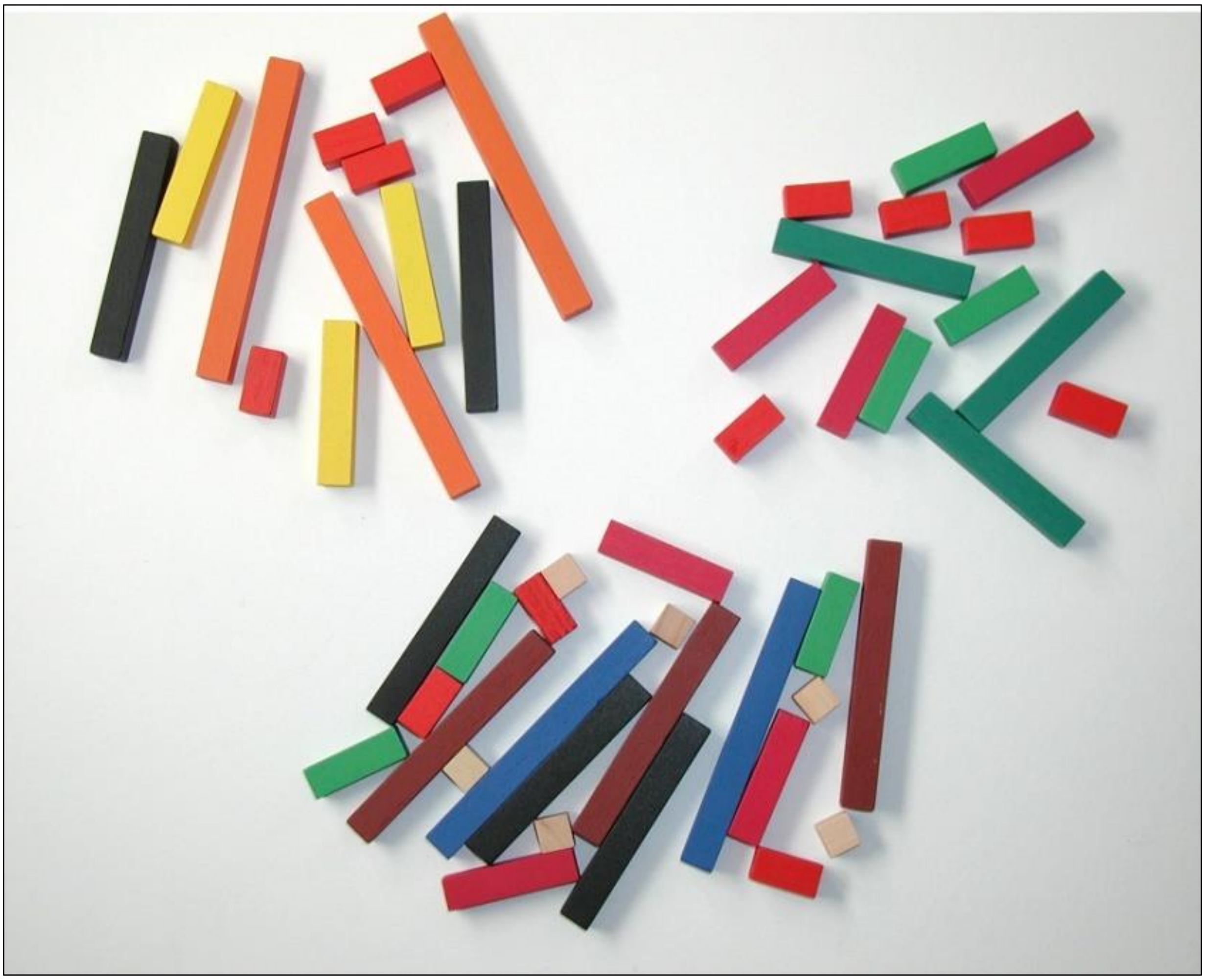
Three handfuls of rods.

This situation displays three such handfuls of rods, and it will be of interest after the study of the last situation to attempt to apply both the methods just described, or variations of these, to the three heaps.



The aim of the exercise is to show the range of mental activity that can be performed from the simple display of a handful of rods. It gives useful practice with number relations that are of value in everyday life.

The game can be varied by asking how many dozens (or how many twenties) the pile contains, instead of tens as described above. In countries where the number twelve is important, this exercise gives a facility that is obviously useful.

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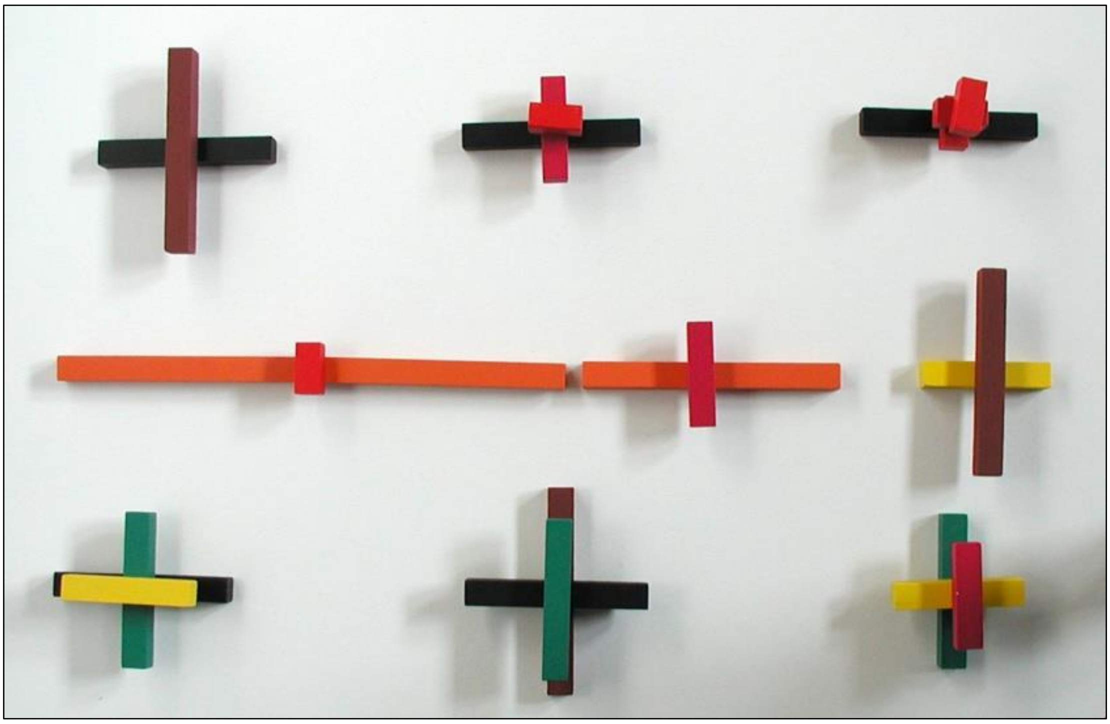
Products, towers and H.C.F.

This situation contains examples of important uses of the Cuisenaire rods for study in the understanding of products, of swift calculations and of the way in which one situation can be turned around in a large number of ways, always producing new information, another notation or new words.



a) The first line shows how one product can be changed into another:

$$7 \times 8 = 7 \times 4 \times 2 = 7 \times 2 \times 2 \times 2$$

A cross thus becomes a tower*. Reading to the left a tower becomes a cross. If the second factor of the cross is changed into a product of prime factors, we obtain the tower of each number that shows only its prime factors.

Thus:
$48 = 6 \times 8 = 3 \times 2 \times 2 \times 2 \times 2$

The decompositions of two numbers into their prime factors will yield two towers showing at once which are the common factors, and which of these is the highest common factor (H. C. F.).

Towers are also the source of powers since the height of the tower can be read as its index: $8 = 2 \times 2 \times 2 = 2^3$, etc.

A new notation appears: where two rods are one horizontal and one vertical, we read the pair as the number representing the horizontal rod taken to the power equal to the number representing the vertical rod. In *Mathematics with Numbers in Colour Book V* this is considered in detail. A first step in the understanding of logarithms can be found here. (This subject is discussed further at the end of the notes to Situation 28.)

The second line shows what happens when we want to read a product using factors not shown in the cross. Thus 40 = 2 x 20 or 4 x 10 or 8 x 5. We can also say that if we double one factor we must divide the other by two. It works both ways. The number 72, for example, which presents more combinations, could now be tried.

The third line shows examples of 'threes', or towers* made of three rods. We can multiply the three numbers together and find the value (*N*) for which the tower stands. We find that the value does not depend on the order of the multiplication; thus in a tower of the three rods *a, b* and *c*.

$N = a \times b \times c$	and also	$N = (a \times b) \times c$
$= a \times c \times b$		$= (a \times c) \times b$
$= c \times b \times a$		$= (c \times b) \times a$
$= b \times a \times c$		$= a \times (c \times b)$
$= b \times c \times a$		$= b \times (a \times c)$
$= c \times a \times b$		$= c \times (a \times b)$

By removing one rod from the tower *a, b, c* we are effectively dividing the number represented by the tower by the number represented by that rod. Thus $\frac{N}{a} = bxc$ indicates that when rod *a* is removed, the number is divided by *a*, the result being what is left, *b x c*. Hence we find that by removing one rod, or a cross:

$\frac{N}{a} = b\,x\,c$ $\quad \frac{N}{b} = a\,x\,c$ $\quad \frac{N}{c} = a\,x\,b$

$\frac{N}{a\,x\,b} = c$ $\quad \frac{N}{b\,x\,c} = a$ $\quad \frac{N}{c\,x\,a} = b$

$\frac{1}{a}x\,N = b\,x\,c$ $\quad \frac{1}{b}\,x\,N = a\,x\,c$ $\quad \frac{1}{c}x\,N = a\,x\,b$

$\frac{1}{a\,x\,b}\,x\,N = c$ $\quad \frac{1}{a\,x\,c}\,x\,N = b$ $\quad \frac{1}{c\,x\,b}x\,N = a$

From this work many other relations can be established such as:

$\frac{1}{a}x\,N + \frac{1}{c\,x\,b}\,x\,N =$

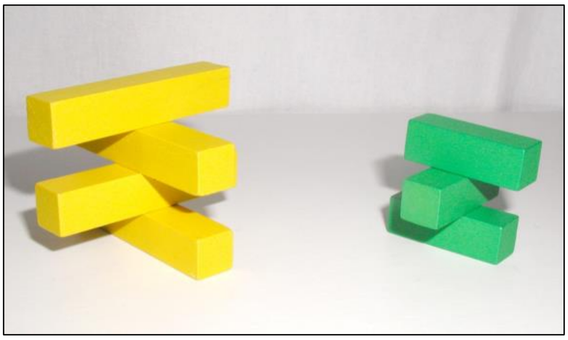
$\frac{1}{b}\,x\,N - \frac{1}{c}\,x\,N =$

n x N (where *n* is an integer) =

Try to solve $\frac{1}{a}\,x\,N + \frac{1}{c\,x\,b}\,x\,N =$

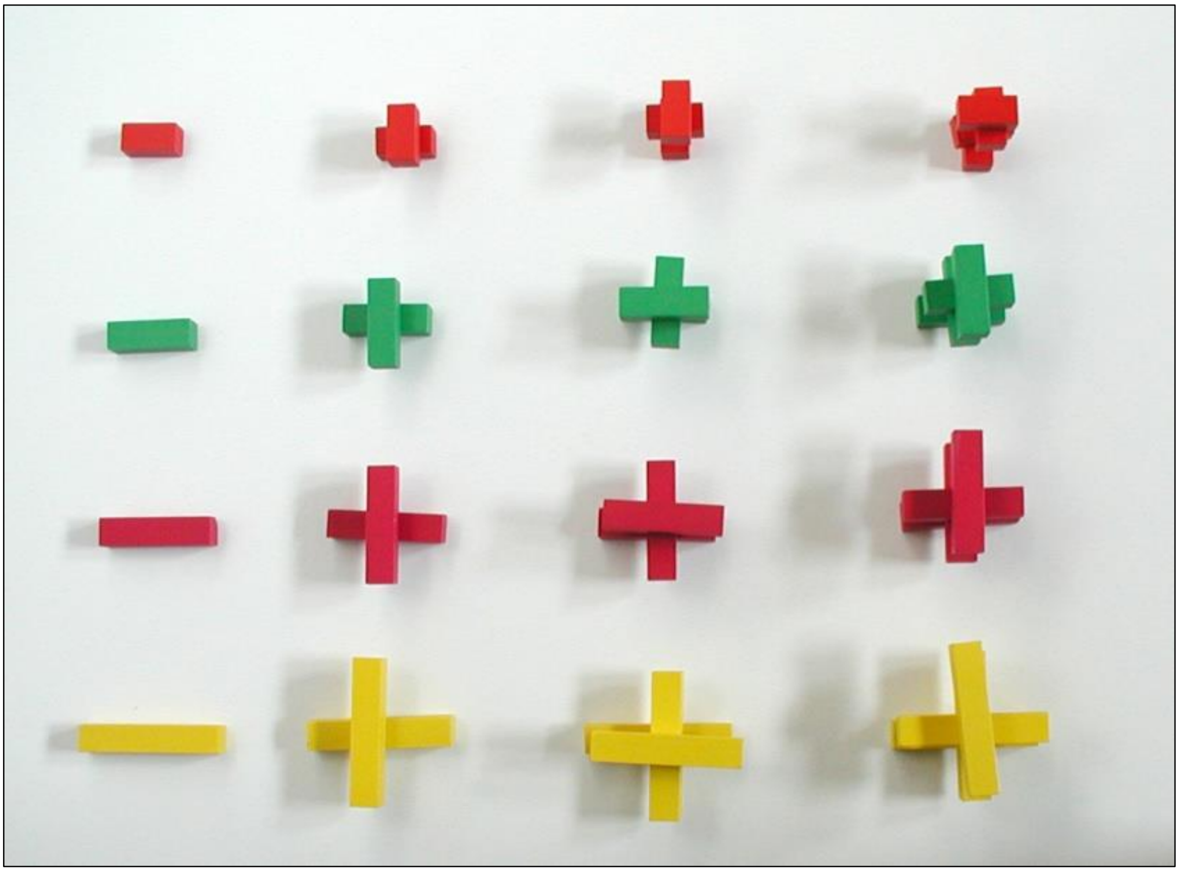
Constant practice with such exercises creates an immense store of arithmetical bonds which will be found useful on many occasions.

* Towers are better illustrated below:-



The yellow tower represents $5 \times 5 \times 5 \times 5$ or 5^4. The light green tower represents $3 \times 3 \times 3$ or 3^3.

Powers as special towers



This situation gives an idea of how special towers are to be considered as <u>powers</u>, and leads to the recognition that the laws of powers are independent of the particular element considered.

This illustration can also serve as groundwork for geometric progressions. See also the notes to Situation 28.

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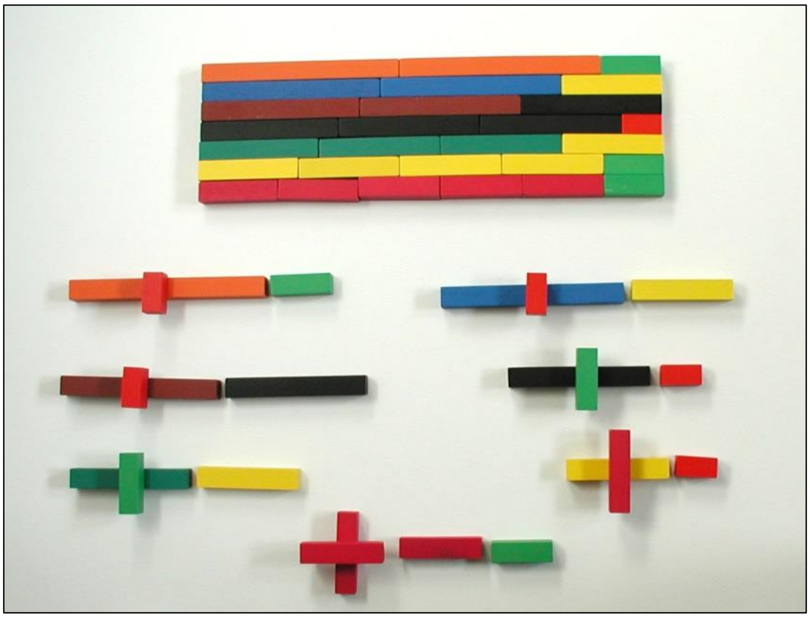
Different scales of numeration



This situation provides a basis for the introduction of different scales of numeration.

In the illustration above the example of 23 is considered, this number being formed in each row using rods of one colour, with the remainder added.

Thus: $23 = 2 \times 10 + 3$
$= 2 \times 9 + 5$
$= 2 \times 8 + 7$
$= 3 \times 7 + 2$
$= 3 \times 6 + 5$
$= 4 \times 5 + 3$
$= 1 \times 4^2 + (1 \times 4) + 3$

The writing of the last expression shows that we do not allow ourselves to employ, using that notation, any figure that represents a number greater than the rod used repeatedly.

A given length can therefore be written in various ways. If we consider the base to be the orange rod, we write the length chosen here as 23.
If the base were the blue rod, we should write 25 - for this means 2 blue rods and a yellow.
27 is the writing for the length in the base 8;
32 (base 7);
35 (base 6);
43 (base 5);
113 (base 4).
Find the expressions in base 3 and base 2.

Once this situation is considered, we should work out examples using other initial lengths in order to gain experience with different bases. Operating on them will become an easy matter since the rules are the same in all bases, even if the writings look different. For example, 7 x 7 is equal to 100 (base 7). In *Mathematics with Numbers in Colour_Book V* these questions are treated for primary school learners.

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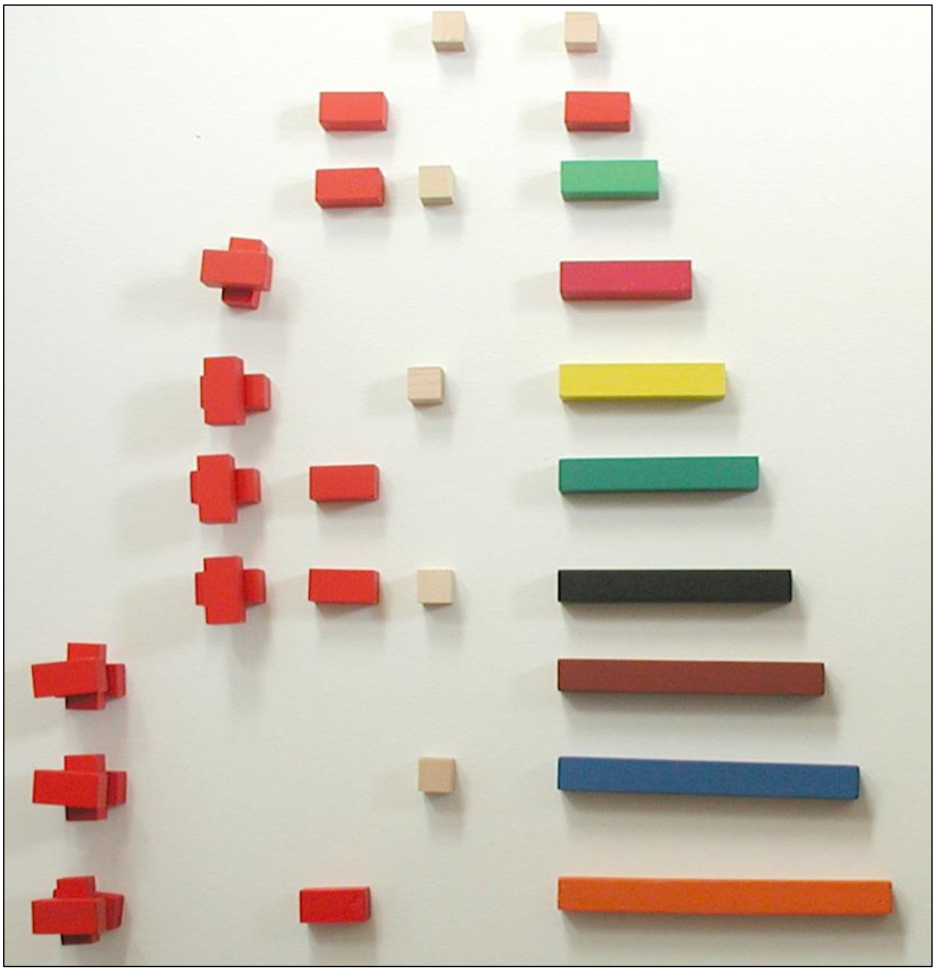
The binary scale



An example of working out the lengths of the rods using only red and white rods appears here. When we reach the pink, we use the red-red cross; and when we reach the tan coloured rod we use the red-red-red – a tower of three red rods.

If we translate each scheme on the left into figures (using 1 to express presence and 0 to express absence) we can write; 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010 as the lengths of the ten rods in the binary scale.

Any length made of rods can be written using this notation and calculations can be performed in it, as in the ordinary denary notation.

If we use light green rods in place of the red, but in conjunction with one red or one white as appropriate, any length of rods can be made up and the corresponding written forms will contain the figures 0, 1, 2.

It is clear that any length of rods can similarly be made up using rods of one colour only with a rod from one of the shorter lengths.

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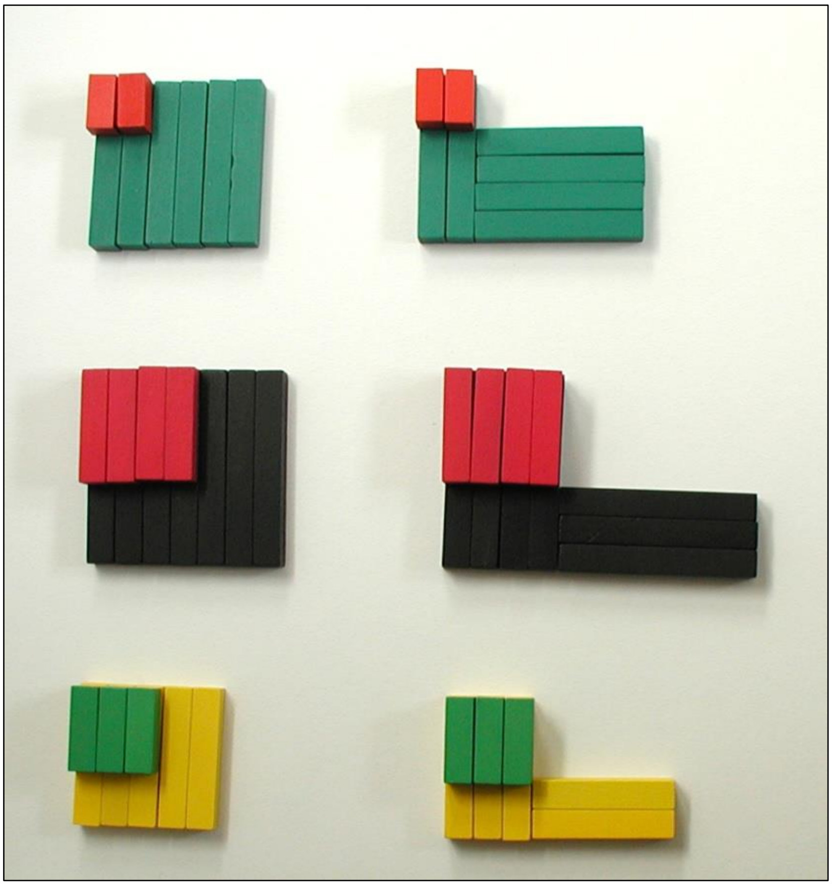
STUDY OF SQUARES

The difference of two squares

The equivalence $a^2 - b^2 = (a + b)(a - b)$



The difference between two squares can be seen by placing the smaller square on the larger, in one corner. When the rods of the larger not covered by the small square are rotated through 90^0, a rectangle is formed whose dimensions are on the one hand the sum of, and on the other the difference between, the lengths of the sides of the two squares.

In the first of the three examples shown, the small red square is placed on the larger dark green square. The dark green rods that are not covered by red rods are then swung round to form a rectangle with the uncovered portion of the remaining dark green rods. The longer dimension of this rectangle measures dark green plus red; the shorter dimension measures dark green minus red. In notation, the difference between those two particular squares is rendered thus:

$$d^2 – r^2 = (d + r)(d – r)$$

What would the notation be for the other two examples illustrated?

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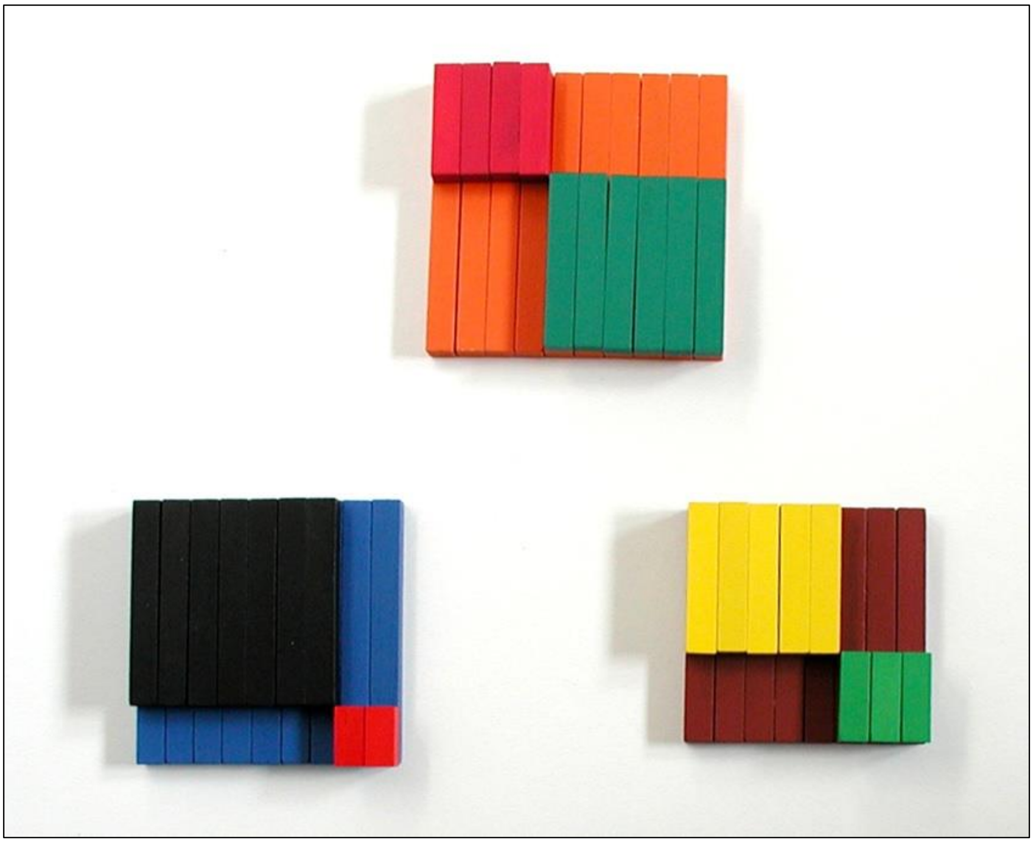
The square of a sum

The equivalence $(a + b)^2 = a^2 + b^2 + 2ab$

This relationship obtains when we compare the square on a sum of two lengths with the sum of the two squares on each length.



We note that the latter taken together are smaller than the first square by two equal rectangles which have for their dimensions one side from each of the two smaller squares.
In the first example, squares built on the lengths of the pink and the dark green rods are placed on a square built on the combined lengths of these two rods. In the illustration this third square is formed with orange rods since the pink and the dark green rods end to end are equivalent in length to an orange rod.

It can be seen that the area of the square of orange rods is equivalent to that of the combined areas of the two smaller squares, plus the areas of the two uncovered rectangles defined by the sides of the smaller squares.

In this example the relationship can be expressed as:

$$(d + p)^2 = d^2 + p^2 + 2(d \times p)$$

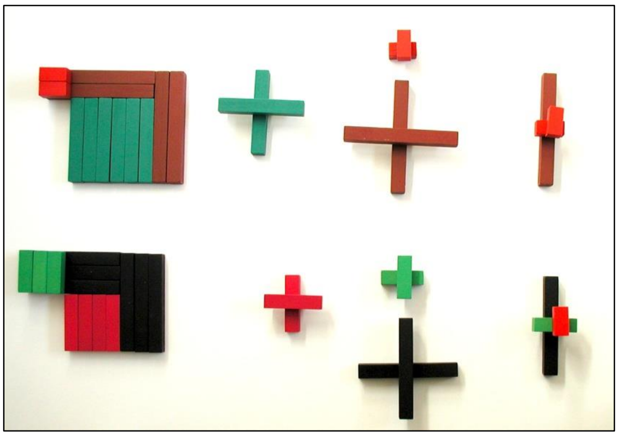
What would the notation be for the other two examples illustrated?

***

The square of difference

The equivalence $(a - b)^2 = a^2 + b^2 - 2ab$

The square of a difference is harder to perceive but is, all the same, easily produced with the rods. The details of the actions needed to establish it appear in-the illustration.



It is necessary to note that a square can always be represented as the square of a difference, for example:

$$\begin{aligned} 6^2 &= (8 - 2)^2 \\ &= (10 - 4)^2 \\ &= (12 - 6)^2 \\ &= (13 - 7)^2 \end{aligned}$$

So a dark green square could be compared with any number of related pairs of squares.

If we border the dark green square on two sides with tan rods and attempt to form a new square, we find that two tan rods are needed on one side and two on the other and that this actually produces an area equal to $8^2 + 2^2$. Bordering the dark green square with orange rods in similar fashion, we should need four on one side and four on the other, producing an area equal to $10^2 + 4^2$. But if we remove the borders from these new figures, we have to subtract in the first case 2 x (2 x 8), and in the second 2 x (4 x 10), re-obtaining both times the dark green square. So in this example,

firstly, the dark green square (or 6^2) can be considered as $(8 - 2)^2$;
secondly if we add two tan coloured rectangles each 2 x 8 to the dark green square we increase the size of the figure to 8^2+2^2;
thirdly, we can return to the original square by removing the two added rectangles.

Consequently we can write that:

$$8^2 + 2^2 - 2 \times (2 \times 8) = (8 - 2)^2$$

or in terms of the actual rods used in the example:

$$(t - r)^2 = t^2 + r^2 - 2(t \times r)$$

In the lower example we see that:

$$4^2 = (7 - 3)^2 = 7^2 + 3^2 - 2 \times (3 \times 7)$$

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